Spaces of curves on manifolds

Our goal in these notes is to provide an extensive discussion of spaces of curves on manifolds especially with an eye towards applications in semi-Riemannian geometry, and especially Lorentzian geometry. There is a plethora of possible choices both of regularity classes of curves and of topologies on suitable sets of such curves. We shall focus here mostly on the case of continuous curves with topologies closely related to the so-called *compact-open* topologies. This choice is motivated by its large applicability.

1 Prelude: the compact-open topology on spaces of continuous maps

We digress here from our main exposition to review a number of important facts about the so-called *compact-open* topology on spaces of continuous maps which are fundamental for us here. Readers familiar with these facts or unwilling to go too deep in them at this moment should just skim over the main results and definitions in this section order to fix notation and terminology, and then proceed to the next section.

Given two topological spaces X and Y, let $C(X,Y)$ denote the set of all continuous maps from X into Y. Given a compact subset $K \subset X$ and an open subset $\mathcal{U} \subset Y$, let $V(K,\mathcal{U})$ denote the set of all functions $f \in C(X,Y)$ such that $f(K) \subset \mathcal{U}$. The topology generated by the collection of all such $V(K,\mathcal{U})$ is called the *compact-open topology* on $C(X,Y)$.

We use $C_c(X,Y)$ as the shorthand for the *pair* (i.e., the topological space) formed by $C(X,Y)$ endowed with the compact-open topology. Our main goal here will be to give suitable conditions on both X and Y to establish the all-important *Arzelà-Ascoli theorems*, which characterize in a convenient fashion the (pre)compact subsets of $C_c(X,Y)$.

When the topology of Y is metrizable, the following well-known result applies. For this reason, the compact-open topology is also referred to as the topology of *uniform convergence in compact subsets* in this context.

Theorem 1.1 *Let X, Y be topological spaces with Y metrizable, and pick any metric d generating the topology on Y. Then, given a sequence $(f_k) \subset C(X,Y)$ and $f \in C(X,Y)$,*

$$f_k \overset{C_c(X,Y)}{\longrightarrow} f \Leftrightarrow f_k|_K \to f|_K \text{ } d\text{-uniformly for any compact subset } K \subset X.$$

Proof. ($\Rightarrow$)
Suppose $f_k \overset{C_c(X,Y)}{\longrightarrow} f$, and let $K \subset X$ be a compact set. Denote an open ball in (Y,d) of radius $r > 0$ and center $y \in Y$ by $B^d_r(y)$, and the closed ball of same radius and center by $\overline{B}^d_r(y)$.

Fix $\varepsilon > 0$. Since f is continuous, for each $x \in K$ we can pick some $U_x \ni x$ open subset of X for which $f(U_x) \subset B^d_{\varepsilon/3}(f(x))$. Then, it is easily checked that $f(\overline{U_x}) \subset \overline{B}^d_{\varepsilon/3}(f(x)) \subset B^d_{\varepsilon/2}(f(x))$.

Now, $\{U_x\}_{x\in K}$ is an open cover of K, from which we extract a finite subcover $\{U_{x_1}, \dots, U_{x_\ell}\}$. Note that each $\overline{U_{x_i}} \cap K$ $(i = 1, \dots, \ell)$ is closed and contained in K, hence also compact. The set

$$\mathcal{V} := \bigcap_{i=1}^{\ell} V(\overline{U_{x_i}} \cap K, B^d_{\varepsilon/2}(f(x_i))$$

is an open set in $C_c(X, Y)$ containing f. Using the convergence of (f_k) in the compact-open topology, there exists some $N \in \mathbb{N}$ for which $f_k \in \mathcal{V}$ whenever $k \in \mathbb{N}$ is larger than N. Let $z \in K$, and $i \in \{1, \dots, \ell\}$ for which $z \in U_{x_i}$. Then, $z \in \overline{U_{x_i}} \cap K$, and hence $d(f_k(z), f(x_i)) < \varepsilon/2$ for $k > N$ by our choices. Thus, using the triangle inequality we have

$$d(f_k(z), f(z)) \leq d(f_k(z), f(x_i)) + d(f(x_i), f(z)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whenever $k > N$. Since z was chosen arbitrarily we conclude that $f_k|_K \to f|_K$ d-uniformly as desired.

$(\Leftarrow)$

Suppose now $f_k|_K \to f|_K$ d-uniformly for any compact subset $K \subset X$, and fix any $C_c(X, Y)$-open set $\mathcal{U} \ni f$. Then there exist compact subsets $K_1, \dots, K_\ell$ of X and open subsets $U_1, \dots, U_\ell$ of Y for which

$$f \in \bigcap_{i=1}^{\ell} V(K_i, U_i) \subset \mathcal{U}.$$

For each $i \in \{1, \dots, \ell\}$ and each $x \in K_i$, choose a number $\varepsilon^i_x > 0$ for which

$$B^d_{2\varepsilon^i_x}(f(x)) \subset U_i.$$

As $\{B^d_{\varepsilon^i_x}(f(x))\}_{x\in K_i}$ is an open covering of the compact set $f(K_i)$, choose a finite subcover

$$\{B^d_{\varepsilon^i_{x^i_1}}(f(x^i_1)), \dots, B^d_{\varepsilon^i_{x^i_{j_i}}}(f(x^i_{j_i}))\}.$$

Let $\varepsilon := min\{\varepsilon^i_l \, : \, i = 1, \dots, \ell; l = 1, \dots, j_i\}$. Let $i \in \{1, \dots, \ell\}$ be given. Using uniform convergence in K_i choose $N_i \in \mathbb{N}$ such that $d(f_k(x), f(x)) < \varepsilon$ for every $x \in K_i$ whenever $k \in \mathbb{N}$ is larger than N_i. For each $z \in K_i$, there exists some $x^i_{j_z}$ with $f(z) \in B^d_{\varepsilon^i_{x^i_{j_z}}}(f(x^i_{j_z}))$ for some $j_z \in \{1, \dots, j_i\}$. Thus, for $k > N_i$, the triangular inequality gives

$$d(f_k(z), f(x^i_{j_z})) \leq d(f_k(z), f(z)) + d(f(z), f(x^i_{j_z})) < \varepsilon + \varepsilon^i_{x^i_{j_z}} \leq 2\varepsilon^i_{x^i_{j_z}},$$

that is, $f_k(z) \in B^d_{2\varepsilon^i_{x^i_{j_z}}}(f(x^i_{j_z})) \subset U_i$, whence we conclude that $f_k(K_i) \subset U_i$ for $k > N_i$. Finally, set $N = max\{N_1, \ldots, N_\ell\}$. Then for any $k \in \mathbb{N}$ larger than N we will have $f_k \in \bigcap_{i=1}^{\ell} V(K_i, U_i) \subset \mathcal{U}$. Thus, $f_k \to f$ in $C_c(X, Y)$ and the proof is complete.

□

We shall be actually interested exclusively in the case when Y is metrizable (indeed when it is a manifold), so that the previous result applies. In this context, if X is also "nice", then so is $C_c(X, Y)$. We deal with two such situations here: when X is compact (but not necessarily Hausdorff), and when X is Hausdorff, second countable and locally compact (which of course occurs, e.g., when X is a finite-dimensional smooth manifold). It is for these two cases that we shall prove Arzelà-Ascoli theorems.

1.1 Arzelà-Ascoli I: the compact case

Theorem 1.2 *Let X, Y be topological spaces with X compact and Y metrizable. Fix a metric d on Y whose associated topology coincides with that of Y. Then, the following statements hold.*

i) The map

$$d_\infty : (f_1, f_2) \in C(X, Y) \times C(X, Y) \mapsto \sup_{x \in X} d(f_1(x), f_2(x)) \in \mathbb{R}_+$$

is a metric on $C(X, Y)$.

ii) If (Y, d) is a complete metric space, then so is $(C(X, Y), d_\infty)$.

iii) The metric topology on $C(X, Y)$ associated with d_∞ coincides with the compact-open topology.

Proof. Proving (i) is easy and left as an exercise for the reader.
(ii)
Suppose (Y, d) is complete and let $(f_k)_{k \in \mathbb{N}}$ be a Cauchy sequence on the metric space $(C(X, Y), d_\infty)$. Then for each $x \in X$ the sequence $(f_k(x))_{k \in \mathbb{N}}$ is Cauchy in (Y, d) (check this!) so the function

$$f : x \in X \mapsto \lim_{k \to +\infty} f_k(x) \in Y$$

is well-defined. We then need to check that $f \in C(X, Y)$ and then that $f_k \overset{d_\infty}{\rightrightarrows} f$. We establish both of these facts via similar arguments.

Fix $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that

$$k, k' \in \mathbb{N}, k, k' \geq k_0 \Longrightarrow d_\infty(f_k, f_{k'}) < \varepsilon/4.$$

By the definition of f we can also choose, for each $x \in X$, a $k_x \in \mathbb{N}$ larger than k_0 such that $d(f_{k_x}(x), f(x)) < \varepsilon/4$.

To prove the continuity of f, let $x_0 \in X$. Since $f_{k_{x_0}}$ is in particular continuous at x_0, there exists an open subset $U \ni x_0$ of X for which

$$x \in U \Longrightarrow d(f_{k_{x_0}}(x), f_{k_{x_0}}(x_0)) < \varepsilon/4.$$

Therefore, for any $x \in U$ we have, repeatedly using the triangle inequality,

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_{k_x}(x)) + d(f_{k_x}(x), f_{k_{x_0}}(x)) \\ &+ d(f_{k_{x_0}}(x), f_{k_{x_0}}(x_0)) + d(f(x_0), f_{k_{x_0}}(x_0)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + d_\infty(f_{k_x}, f_{k_{x_0}}) + \frac{\varepsilon}{4} \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

where we have used that $k_x, k_{x_0} > k_0$ to obtain the last inequality. Thus, f is continuous at x_0 and we conclude that $f \in C(X, Y)$, as desired.

To establish convergence, note that for any $k > k_0$ we have, for every $x \in X$,

$$\begin{aligned} d(f_k(x), f(x)) &\leq d(f_k(x), f_{k_x}(x)) + d(f_{k_x}(x), f(x)) \\ &< \frac{\varepsilon}{4} + d_\infty(f_{k_x}, f_k) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \end{aligned}$$

so

$$d_\infty(f_k, f) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus, $f_k \overset{d_\infty}{\rightrightarrows} f$. We conclude that $(C(X, Y), d_\infty)$ is indeed complete.

(iii)

Let $\mathcal{U} \subset C(X, Y)$ be open in the compact-open topology and fix $f \in \mathcal{U}$. Then, there exist compact subsets $K_1, \ldots K_m$ of X and open subsets $U_1, \ldots, U_m$ of Y for which

$$f \in \bigcap_{i=1}^{m} \mathcal{V}(K_i, U_i) \subset \mathcal{U}.$$

Fix $i \in \{1, \ldots, m\}$. For each $x \in K_i$, choose $\varepsilon_x^i > 0$ such that $B^d_{2\varepsilon_x^i}(f(x)) \subset U_i$. Since $\{B^d_{\varepsilon_x^i}(f(x))\}_{x \in K_i}$ is an open cover othe compact set $f(K_i) \subset Y$, extract a finite subcover

$$\{B^d_{\varepsilon^i_{x^i_1}}(f(x^i_1)), \ldots, B^d_{\varepsilon^i_{x^i_{k_i}}}(f(x^i_{k_i}))\}.$$

Finally, let $\varepsilon_i := \min\{\varepsilon^i_{x^i_1}, \ldots, \varepsilon^i_{x^i_{k_i}}\}$. Given $\hat{f} \in B^{d_\infty}_{\varepsilon_i}(f)$, and $x \in K_i$, let $j \in \{1, \ldots, k_i\}$ such that $f(x) \in B^d_{\varepsilon^i_{x^i_j}}(f(x^i_j))$. Then

$$d(\hat{f}(x), f(x^i_j)) \leq d(\hat{f}(x), f(x)) + d(f(x), f(x^i_j)) < d_\infty(\hat{f}, f) + \varepsilon^i_{x^i_j} < \varepsilon_i + \varepsilon^i_{x^i_j} \leq 2\varepsilon^i_{x^i_j},$$

whence $\hat{f}(x) \in B^d_{2\varepsilon^i_{x^i_j}}(f(x^i_j)) \subset U_i$. We conclude that $\hat{f} \in \mathcal{V}(K_i, U_i)$. Since this reasoning is valid for each i, we may now set $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_m\}$, and conclude that

$$B^{d_\infty}_\varepsilon(f) \subset \bigcap_{i=1}^m \mathcal{V}(K_i, U_i) \subset \mathcal{U},$$

and thus that $\mathcal{U}$ is indeed open in $(C(X,Y), d_\infty)$.

Now, fix $f \in C(X,Y)$, and a number $r > 0$. By the continuity of f and the compactness of X we can choose a finite open cover $\mathcal{O}_1, \ldots, \mathcal{O}_m$ of X and elements $x_i \in \mathcal{O}_i$ $(i = 1, \ldots, m)$ such that

$$f(\mathcal{O}_i) \subset B^d_{r/4}(f(x_i)) \quad \forall i \in \{1, \ldots, m\}.$$

Note also that again by continuity, for each $i \in \{1, \ldots, m\}$ we have $f(\overline{\mathcal{O}_i}) \subset \overline{B}^d_{r/4}(f(x_i))$, where the latter denotes the closed ball in Y of radius $r/4$. Since $K_i := \overline{\mathcal{O}_i}$ is closed in the compact set X, it is also compact. Let

$$\hat{f} \in \bigcap_{i=1}^m \mathcal{V}(K_i, B^d_{r/4}(f(x_i))).$$

Given $x \in X$, pick $j \in \{1, \ldots, m\}$ with $x \in \mathcal{O}_j \subset K_j$. Then

$$d(\hat{f}(x), f(x)) \leq d(\hat{f}(x), f(x_j)) + d(f(x), f(x_j)) < \frac{r}{4} + \frac{r}{4} = \frac{r}{2}.$$

Since x is arbitrary, $\hat{f} \in B^{d_\infty}_r(f)$. We conclude that

$$\bigcap_{i=1}^m \mathcal{V}(K_i, B^d_{r/4}(f(x_i))) \subset B^{d_\infty}_r(f),$$

and thus that $B^{d_\infty}_r(f)$ is open in $C_c(X,Y)$. This concludes the proof.

□

Of course, in the conditions of Theorem 1.2 and in view of Theorem 1.1, a sequence (f_k) converges in the compact-open topology if and only if its converges uniformly on X.

Recall that if (M,d) is a metric space and $A \subset M$ is any set, then A is said to be *totally bounded* if for any $\varepsilon >$ there exist $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in M$ such that

$$A \subset \bigcup_{i=1}^m B^d_\varepsilon(x_i).$$

Recall also that A *is compact if and only if it is totally bounded and complete* (as a metric space with the restricted metric). In particular, this means that if (M,d) is a *complete* metric space, then A is precompact (i.e., $\overline{A}$ is compact) if and only if A is totally bounded.

Theorem 1.3 (Arzelà-Ascoli - the compact case) *Let X be a compact topological space and let (Y,d) be a* complete *metric space. A subset $E \subset C(X,Y)$ is precompact in the compact-open topology if and only if it satisfies the following two conditions.*

i) E is equicontinuous*, i.e., for any $x \in X$ and any $\varepsilon > 0$ there exists an open set $U \ni x$ of X such that*

$$f \in E, y \in U \Longrightarrow d(f(y), f(x)) < \varepsilon.$$

ii) E is pointwise precompact*, i.e., for every $x \in X$ the set*

$$E(x) := \{f(x) \, : \, f \in E\}$$

is precompact in Y.

Proof. Let $E \subset C(X,Y)$. In view of Theorem 1.2 and with the notation therein $(C(X,Y), d_\infty)$ is also a complete metric space. Due to the previous remarks, E is precompact if and only if it is totally bounded in $(C(X,Y), d_\infty)$. Thus we actually shall prove:

$$E \text{ is totally bounded if and only if } (i) \text{ and } (ii) \text{ hold.}$$

We fix an abitrary $\varepsilon > 0$ for the rest of the proof.

Assume first that E is totally bounded. Pick $f_1, \ldots, f_m \in C(X,Y)$ such that $E \subset \cup_{i=1}^m B_{\varepsilon/3}^{d_\infty}(f_i)$. Let $x_0 \in X$. By continuity, we can choose an open set $U \ni x_0$ of X for which

$$x \in U \Longrightarrow d(f_i(x), f_i(x_0)) < \varepsilon/3, \quad \forall i \in \{1, \ldots, m\}.$$

Let $f \in E$ be given. Pick $j \in \{1, \ldots, m\}$ for which $f \in B_{\varepsilon/3}^{d_\infty}(f_i)$. Then

$$x \in U \Rightarrow d(f(x), f(x_0)) \leq d(f(x), f_j(x)) + d(f_j(x), f_j(x_0)) + d(f_j(x_0), f(x_0)) < 2d_\infty(f, f_j)\varepsilon/3 < \varepsilon.$$

Besides, we immediately have

$$d(f(x_0), f_j(x_0)) \leq d_\infty(f, f_j) < \varepsilon/3 < \varepsilon,$$

so $f(x_0) \in B^d(f(_j(x_0))$. Since $f \in E$ and x_0 were chosen arbitrary, we now conclude that E is equicontinuous, and

$$E(x_0) \subset \cup_{i=1}^m B_\varepsilon^d(f_i(x_0)),$$

thus $E(x_0)$ is totally bounded in (Y,d), hence precompact therein.

Now assume that (i) and (ii) hold. Using equicontinuity and the compactness of X we can choose a finite open cover $\{U_1, \ldots, U_m\}$ of X and $\{x_1, \ldots, x_m\} \subset X$ such that for each $i = 1, \ldots, m$,

$$x \in U_i \Longrightarrow d(f(x), f(x_i)) < \varepsilon/6, \quad \forall f \in E. \tag{1}$$

Using (ii) we can easily check that

$$C^m := \{(f(x_1), \dots, f(x_m)) \,:\, f \in E\}$$

is precompact and hence totally bounded in the (complete) metric space (Y^m, d^m) where d_m is the metric given by

$$d^m((z_1, \dots, z_m), (y_1, \dots, y_m)) = \sum_{j=1}^{m} d(z_j, y_j).$$

Therefore, there exist $z^1, \dots, z^\ell \in Y^m$ such that

$$C^m \subset \bigcup_{j=1}^{\ell} B^{d_m}_{\varepsilon/6}(z^j). \tag{2}$$

In addition, we can assume, without loss of generality, that $B^{d_m}_{\varepsilon/6}(z^j) \cap C^m \neq \emptyset$, so we pick $f_1, \dots, f_\ell \in E$ such that

$$(f_i(x_1), \dots, f_i(x_m)) \in B^{d_m}_{\varepsilon/6}(z^i), \quad \forall i \in \{1, \dots, \ell\}. \tag{3}$$

Now, let $f \in E$ be given. Since $(f(x_1), \dots, f(x_m)) \in C^m$, by (2) there exists $\lambda \in \{1, \dots, \ell\}$ such that $(f(x_1), \dots, f(x_m)) \in B^{d_m}_{\varepsilon/6}(z^\lambda)$. But then, by (3),

$$\sum_{i=1}^{m} d(f(x_i), f_\lambda(x_i)) \leq \sum_{i=1}^{m} [d(f(x_i), z_i^\lambda) + d(z_i^\lambda, f_\lambda(x_i))] < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}. \tag{4}$$

Then, given $x \in X$, we have $j \in \{1, \dots, m\}$ for which $x \in U_j$, and we have by the triangle inequality

$$\begin{aligned} d(f(x), f_\lambda(x)) &\leq d(f(x), f(x_j)) + d(f(x_j), f_\lambda(x_j)) + d(f_\lambda(x_j), f_\lambda(x)) \\ &\overset{(1)}{<} \frac{\varepsilon}{3} + d(f(x_j), f_\lambda(x_j)) \leq \frac{\varepsilon}{3} + \sum_{i=1}^{m} d(f(x_i), f_\lambda(x_i)) \\ &\overset{(4)}{<} \frac{2\varepsilon}{3}, \end{aligned}$$

whence we conclude that

$$d_\infty(f, f_\lambda) \leq \frac{2\varepsilon}{3} < \varepsilon,$$

and hence that

$$E \subset \bigcup_{\lambda=1}^{\ell} B^{d_\infty}_{\varepsilon}(f_\lambda).$$

That its, E is totally bounded as desired.

$\square$

1.2 Arzelà-Ascoli II: the locally compact case

The previous compactness assumption on X is too restrictive for most of our applications. Hence we weaken it as follows.

Theorem 1.4 *Let X, Y be topological spaces with X locally compact, Hausdorff and second-countable, and Y metrizable. Then the compact-open topology on $C(X, Y)$ is metrizable. If Y is in addition* completely metrizable*, i.e., if it admits a* complete *topological metric, then so does $C_c(X, Y)$.*

Proof. Fix throughout the proof a metric d on Y whose associated topology coincides with that of Y. We proceed by a series of Claims.
Claim 1: there exists a sequence $(K_m)_{m\in\mathbb{N}}$ of compact subsets of X for which

$$K_m \subset \operatorname{int} K_{m+1}, \quad \forall m \in \mathbb{N}, \tag{5}$$
$$X = \bigcup_{m\in\mathbb{N}} K_n. \tag{6}$$

To prove this, let $\mathcal{B}$ be a countable basis for the topology of X. Let

$$\mathcal{B}' := \{B \in \mathcal{B} : \exists K \subset X \text{ compact such that } \overline{B} \subset K\}.$$

Given $x \in X$, since X is locally compact there exists a compact subset $K \subset X$ with $x \in \operatorname{int} K$. Since $\mathcal{B}$ is a basis, there exists $B \in \mathcal{B}$ with

$$x \in B \subset \operatorname{int} K \subset K,$$

and since X is Hausdorff, K is closed, so $\overline{B} \subset K$. We conclude that $B \in \mathcal{B}'$, and hence that $\mathcal{B}'$ is a countable open cover of X. Choose an enumeration $\mathcal{B}' = \{B_\ell : \ell \in \mathbb{N}\}$.

We now define $(K_m)_{m\in\mathbb{N}}$ inductively as follows. Let $K_1 := \overline{B_1}$, and given a sequence of j (≥ 1) compact subsets $K_1, \ldots, K_j$ satisfying (5) let k_j be the least natural number $\geq j$ for which

$$K_j \subset B_1 \cup \ldots \cup B_{k_j}.$$

Now, define

$$K_{j+1} := \overline{B_1} \cup \ldots \cup \overline{B_{k_j}}.$$

Then we immediately see that $K_j \subset \operatorname{int} K_{j+1}$. In addition, since $\mathcal{B}'$ covers X, given $x \in X$ there exists $\ell \in \mathbb{N}$ for which

$$x \in B_\ell \subset B_1 \cup \ldots \cup B_\ell \subset B_1 \cup \ldots \cup B_{k_\ell} \subset K_{\ell+1}$$

by construction, since $k_\ell \geq \ell$. Thus, we have shown that $(K_m)_{m\in\mathbb{N}}$ thus built indeed satisfies (6), and Claim 1 is proved.
Fix now for the rest of the proof the sequence $(K_m)_{m\in\mathbb{N}}$ of compact subsets of X as in the previous Claim. For each $i \in \mathbb{N}$ and any $f_1, f_2 \in C(X, Y)$ define

$$d_i(f_1, f_2) := \sup_{x\in K_i} d(f_1(x), f_2(x)).$$

The proof of the next Claims will rely on the fact that the function

$$\varphi : x \in [0, +\infty) \mapsto \frac{x}{1+x}$$

is an increasing homeomorphism onto $[0, 1)$.
Claim 2: The map

$$D_\infty : (f_1, f_2) \in C(X,Y) \times C(X,Y) \mapsto \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(f_1, f_2)}{1 + d_i(f_1, f_2)} \in \mathbb{R}_+ \tag{7}$$

is a metric on $C(X,Y)$.
It is immediate that D_∞ is a well-defined map, and symmetry is also clear. If $f_1, f_2 \in C(X,Y)$ are such that $D_\infty(f_1, f_2) \equiv 0$, then for each $i \in \mathbb{N}$ we have $d_i(f_1, f_2) \equiv 0$, whence we conclude that $f_1|_{K_i} \equiv f_2|_{K_i}$, and from (6) we conclude that $f_1 \equiv f_2$. Thus all that remains to be shown is the triangle inequality.

To this end, let $f_1, f_2, f_3 \in C(X,Y)$ be given. The triangle inequality for d on Y easily gives, for each $i \in \mathbb{N}$,

$$d_i(f_1, f_2) \leq d_i(f_1, f_3) + d_i(f_3, f_2). \tag{8}$$

Since φ is (strictly) increasing, (8) implies that

$$\frac{d_i(f_1, f_2)}{1 + d_i(f_1, f_2)} \leq \frac{d_i(f_1, f_3) + d_i(f_3, f_2)}{1 + d_i(f_1, f_3) + d_i(f_3, f_2)} \leq \frac{d_i(f_1, f_3)}{1 + d_i(f_1, f_3)} + \frac{d_i(f_3, f_2)}{1 + d_i(f_3, f_2)}. \tag{9}$$

Therefore, multipying both sides of (9) by $1/2^i$ and summing over $i \in \mathbb{N}$ yields

$$D_\infty(f_1, f_2) \leq D_\infty(f_1, f_3) + D_\infty(f_3, f_2)$$

as desired. This concludes the proof of Claim 2.

Claim 3: The metric topology of $(C(X,Y), D_\infty)$ is the same as that of $C_c(X,Y)$.

First, let us show that given $r > 0$ and $f \in C(X,Y)$, the open D_∞-ball $B_r^{D_\infty}(f)$ of radius r and centered at f is open in $C_c(X,Y)$. For that, it is enough to show that there exists compact subsets $C_1, \ldots, C_m \subset X$ and open subsets $U_1, \ldots, U_m$ such that

$$f \in \bigcap_{i=1}^{m} \mathcal{V}(C_i, U_i) \subset B_r^{D_\infty}(f).$$

Using the continuity of φ at 0, pick $\delta > 0$ such that

$$0 \leq x < \delta \Longrightarrow \varphi(x) < \frac{r}{2}.$$

Now, let $N \in \mathbb{N}$ for which

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{r}{2}.$$

Since $K_N \subset X$ is compact and f is continous, we can pick a finite open cover $\{\mathcal{O}_1, \ldots, \mathcal{O}_m\}$ of K_N by precompact subsets and $x_i \in \mathcal{O}_i$ with $f(\mathcal{O}_i) \subset B^d_{\delta/4}(f(x_i))$ $(i = 1, \ldots, m)$. Then, setting $C_i := \overline{\mathcal{O}_i}$ and $U_i := B^d_{\delta/3}(f(x_i))$ for $i \in \{1, \ldots, m\}$, we have

$$f \in \bigcap_{i=1}^{m} \mathcal{V}(C_i, U_i) =: \mathcal{U}.$$

Let $\hat{f} \in \mathcal{U}$. Given $x \in K_N$, let $i \in \{1, \ldots, m\}$ such that $x \in C_i$. Then

$$d(\hat{f}(x), f(x)) \leq d(\hat{f}(x), f(x_i)) + d(f(x_i), f(x)) < \frac{2\delta}{3}.$$

Thus,

$$0 \leq d_1(\hat{f}, f) \leq \cdots \leq d_N(\hat{f}, f) \leq \frac{2\delta}{3} < \delta \Rightarrow \frac{d_\ell(\hat{f}, f)}{1 + d_\ell(\hat{f}, f)} < \frac{r}{2} \quad \ell = 1, \ldots, N.$$

But then

$$D_\infty(\hat{f}, f) = \sum_{\ell=1}^{N} \frac{1}{2^\ell} \frac{d_\ell(\hat{f}, f)}{1 + d_\ell(\hat{f}, f)} + \sum_{\ell=N+1}^{\infty} \frac{1}{2^\ell} \frac{d_\ell(\hat{f}, f)}{1 + d_\ell(\hat{f}, f)} < \frac{r}{2} + \frac{r}{2} < r,$$

i.e. we have established that $\mathcal{U} \subset B^{D_\infty}_r(f)$ as desired.

Conversely, let $\mathcal{U}$ be open in $C_c(X, Y)$ and let $f \in \mathcal{U}$. Since we wish to show that the latter set is open in the metric space $(C(X, Y), D_\infty)$, we only need to show that $C(X, Y) \setminus \mathcal{U}$ is *sequentially closed*, i.e. that for a convergent sequence $f_k \overset{D_\infty}{\to} f$ with $(f_k) \subset C(X, Y) \setminus \mathcal{U}$ we also have $f \in C(X, Y) \setminus \mathcal{U}$.

Fix then such a sequence. Let $\varepsilon > 0$ be given, and fix $\ell \in \mathbb{N}$. Since φ^{-1} is continuous, there exists $0 < \delta_\ell < 1$ such that

$$0 \leq \frac{x}{1 + x} < 2^\ell \delta_\ell \Longrightarrow 0 \leq x < \varepsilon.$$

Let $k_\ell \in \mathbb{N}$ such that for any integer $k > k_\ell$ we have $D_\infty(f_k, f) < \delta_\ell$. Then, for every integer $k > k_\ell$,

$$\frac{d_\ell(f_k, f)}{1 + d_\ell(f_k, f)} \leq 2^\ell D_\infty < 2^\ell \delta_\ell \Longrightarrow d_\ell(f_k, f) < \varepsilon.$$

This shows that

$$f_k|_{K_\ell} \to f|_{K_\ell} \text{ } d\text{-uniformly .}$$

Since ℓ was chosen arbitrarily, given any compact set $K \subset X$, there exists $\ell \in \mathbb{N}$ for which $K \subset K_\ell$ due to (5)-(6), so

$$f_k|_K \to f|_K \text{ } d\text{-uniformly, for any compact set } K \subset X.$$

By Thm. 1.1, $f_k \overset{C_c(X,Y)}{\longrightarrow} f$. If $f \in \mathcal{U}$ then eventually $f_k \in \mathcal{U}$, a contradition. Thus $f \in C(X, Y) \setminus \mathcal{U}$ as desired. Thus, Claim 3 is established.

Claim 4: If (Y, d) is complete, then so is $(C(X, Y), D_\infty)$.

Assume therefore that (Y, d) is complete, and let $(f_k) \subset C(X, Y)$ be a Cauchy sequence in $(C(X, Y), D_\infty)$. For each $\ell \in \mathbb{N}$, arguing simillarly to the last part of the proof of Claim 3 easily establishes that $(f_k|_{K_\ell})$ is Cauchy in the metric space $(C(K_\ell, Y), d_\ell)$. Applying Thm. 1.2(ii) for $X = K_\ell$ we conclude that there exists $f^\ell \in C(K_\ell, Y)$ such that $f_k|_{K_\ell} \to f^\ell$ d-uniformly. Define $f \in C(X, Y)$ by setting

$$f|_{K_\ell} := f^\ell, \quad \forall \ell \in \mathbb{N}.$$

One easily checks that f is well defined, and by construction $f_k|_{K_\ell} \to f|_{K_\ell}$ d-uniformly. By Thm. 1.1, $f_k \overset{C_c(X,Y)}{\longrightarrow} f$, and by Claim 3, $f_k \overset{D_\infty}{\to} f$, thus completing the proof of Claim and of theTheorem.

□

Theorem 1.5 (Arzelà-Ascoli II- the locally compact case) *Let X be a locally compact, Hausdorff and second countable topological space and let (Y, d) be a* complete *metric space. A subset $E \subset C_c(X, Y)$ is precompact if and only if it satisfies the following two conditions.*

i) E is equicontinuous, i.e., for any $x \in X$ and any $\varepsilon > 0$ there exists an open set $U \ni x$ of X such that

$$f \in E, y \in U \Longrightarrow d(f(y), f(x)) < \varepsilon.$$

ii) E is pointwise precompact, i.e., for every $x \in X$ the set

$$E(x) := \{f(x) : f \in E\}$$

is precompact in Y.

Proof. Fix $E \subset C_c(X, Y)$. Just as in Claim 1 in the proof of Thm. 1.4, we fix a sequence $(K_m)_{m\in\mathbb{N}}$ of compact subsets of X satisfying conditions (5)-(6). Fix an integer $\ell \in \mathbb{N}$, and put

$$E_\ell := \{f|_{K_\ell} : f \in E\}.$$

Then $E_\ell \subset C(K_\ell, Y)$. Recall also that the topology on $C(X, Y)$ is given by the complete metric D_∞ defined Thm. 1.4.

Assume first that E is equicontinuous and pointwise compact. Then the same holds (check!) for $E_\ell \subset C(K_\ell, Y)$. By the Arzelà-Ascoli Thm.1.3, E_ℓ is precompact in $C(K_\ell, Y)$. Let $(f_k)_{k\in\mathbb{N}} \subset E$ be given. Then $(f_k|_{K_\ell})_{k\in\mathbb{N}} \subset E_\ell$. Thus, precompactness in this metrizable context implies that there exists $f^\ell \in C(K_\ell, Y)$ such that $f_k|_{K_\ell} \to f^\ell$ d-uniformly. Since these arguments hold for any $\ell \in \mathbb{N}$, define $f \in C(X, Y)$ by setting

$$f|_{K_\ell} := f^\ell, \quad \forall \ell \in \mathbb{N}.$$

One then checks that f is well defined, and by construction $f_k|_{K_\ell} \to f|_{K_\ell}$ d-uniformly. By Thm. 1.1, $f_k \stackrel{C_c(X,Y)}{\longrightarrow} f$, thus proving that E is precompact.

Conversely, assume E is precompact. Then for each $\ell \in \mathbb{N}$, E_ℓ as defined above will be precompact in $C(K_\ell, Y)$. But given any $x_0 \in X$, we have that $x_0 \in \operatorname{int} K_\ell$ for large enough ℓ. Again by Thm.1.3 applied to this K_ℓ, we conclude that E_ℓ is equicontinuous and pointwise compact in K_ℓ. This in turn shows that E is equicontinuous and pointwise compact in X. The details of the arguments are left to the reader.

□

The following very useful Corollary can now be immediately deduced.

Corollary 1.1 *Let N, M be finitely-dimensional smooth manifolds, and h a complete Riemannian metric on M with distance function d_h.*
Let $(f_k : N \to M)_{k\in\mathbb{N}}$ be sequence of continuous functions. Suppose

i) $\{f_k : k \in \mathbb{N}\}$ *is equicontinuous, and*

ii) for every $x \in N$ *the set* $\{f_k(x) : k \in \mathbb{N}\} \subset M$ *is* d_h*-bounded.*

Then there exist a subsequence $(f_{k_i} : N \to M)_{i\in\mathbb{N}}$ and a continuous function $f : N \to M$ such that

$$f_{k_i}|_K \to f|_K \ d_h\text{-uniformly, for every compact set } K \subset N.$$

Comment on the proof. Just note that due to the Hopf-Rinow theorem, as h is complete, d_h-boundedness is the same as precompactness in M, so we can apply Thm. 1.5 to $E = \{f_k : k \in \mathbb{N}\}$.

□

Remark 1.1 A prime example in which Conditions (i) and (ii) in Corollary 1.1 (in the presence of the other assumptions) are met is when there exist some metric d on N (say, arising from a Riemannian metric), a constant $C > 0$ and some $x_0 \in N$ for which

i') $d_h(f_k(x), f_k(y)) \le C \cdot d(x, y)$, $\forall x, y \in N$, $\forall k \in \mathbb{N}$, and

ii') $\{f_k(x_0) : k \in \mathbb{N}\} \subset M$ is d_h-bounded.

(Exercise: Check this.) Condition (i') says the functions are Lipschitz with a"uniform-in-k" Lipschitz constant. This arises in concrete situations, for example, when the functions are C^1 with uniformly bounded derivatives.

2 The space of curves on a manifold

It is the purpose of this section is to define a suitable topology on the space of (continuous) curves on a manifold which is relevant for Lorentzian geometry, and highlight some properties of the corresponding topology on the subset of *continuous causal curves* on a Lorentzian manifold with respect to the *Lorentzian length functional* we shall define below.

We fix throughout a connected n-dimensional smooth (i.e., C^∞) manifold[1] M ($n \geq 2$). We also fix an auxiliary *complete* Riemannian metric h on M, whose distance function we denote by d_h. We denote by I any arbitrary (i.e., closed, semi-closed or open) non-empty interval in $\mathbb{R}$, unless otherwise stated.

2.1 Basic definitions & notation

By a *(parametrized C^0) curve* on M we mean any *continuous* map $\alpha : I \subset \mathbb{R} \to M$. In the specific case when I is a *compact* non-empty interval $I = [a, b]$, α is said to be a (parametrized) curve *segment*. (We shall often omit the qualification "segment" when speaking about such curves, unless we feel that special emphasis is desirable.)

Given a curve $\alpha : I \subset \mathbb{R} \to M$, a point $p \in M$ is said to be a *right [resp. left] endpoint* (of α) if for any neighborhood $U \ni p$, there exists some $t_0 \in I$ for which $\alpha(t) \in U$ for every $t \in I, t \geq t_0$ [resp. $t \leq t_0$. Since M is Hausdorff, a right [resp. left] endpoint of α is unique if it exists. In this case, α is said to be *right-[resp. left-]extendible.* Otherwise it is *right-[resp. left-]inextendible.* If α is both right- and left-inextendible, it is said to be *inextendible.* In particular, any curve segment $\alpha : [a, b] \to M$ is both right- and left-extendible in this sense and $\alpha(a), \alpha(b)$ are its left and right endpoints, respectively.

We collect all curve segments in the set

$$\tilde{\mathcal{C}}_M := \{\alpha : [a, b] \to M \ : \ a, b \in \mathbb{R}, a < b, \ \alpha \text{ is a curve}\},$$

which we refer to as the *space*[2] *of parametrized (C^0) curves* on M.

In the previous definition, curves may differ as maps but have the same image, in which case one sometimes wishes to refer to them as different *parametrizations* of the "same" curve. The point of this terminology is that in some contexts, these parametrizations are in themselves without geometric significance[3].

We can make this geometric invariance more precise as follows. Given curves $\beta : J \to M$ and $\alpha : I \to M$, we say that β is a *reparametrization* of α if there exists an *increasing* homeomorphism $f : J \to I$ with $\beta = \alpha \circ f$.

[1] Here and hereafter, "manifold" means a *finite-dimensional real smooth Hausdorff second countable manifold.*

[2] The apellation "space" is somewhat abusive here as we have only a set without any further structure. But we will soon add a suitable topology thereon. When we do, it will become a *bona fide* topological space, referred to by the same name.

[3] An exception to this general rule are *geodesics* when we have some background affine connection: in this case affine reparametrizations are almost exclusively adopted.

Clearly, "being a reparametrization of" defines an equivalence relation $\sim$ on $\tilde{\mathcal{C}}_M$ by

$$\beta \sim \alpha \Leftrightarrow \exists \text{ an increasing homeomorphism } f : [c,d] \to [a,b] \text{ with } \beta = \alpha \circ f.$$

The *space of (unparametrized C^0) curves* on M is the quotient

$$\mathcal{C}_M := \tilde{\mathcal{C}}_M / \sim .$$

We shall denote by $[\alpha]$ the equivalence class of $\alpha \in \tilde{\mathcal{C}}_M$, although we sometimes abuse notation and omit the brackets if there is no risk of confusion[4].

It will also be convenient to introduce a separate notation for the set of all compact non-degenerate intervals if $\mathbb{R}$:

$$\mathcal{J} = \{[a,b] \subset \mathbb{R} \, : \, a < b\}. \tag{10}$$

We also turn $\mathcal{J}$ into a *metric space* by defining a distance $D_H : \mathcal{J} \times \mathcal{J} \to \mathbb{R}_+$ by

$$D_H([a,b],[c,d]) := |a-c| + |b-d|, \quad \forall [a,b],[c,d] \in \mathcal{J}. \tag{11}$$

Exercise 2.1 (Hausdorff metric) *Given any metric space (X,d), let*

$$\Theta_X := \{K \subset X \, : \, K \textit{ is compact and non-empty}\}.$$

Define the map

$$\rho_X : (A,B) \in \Theta_X \times \Theta_X \mapsto max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\} \in \mathbb{R}.$$

a) Show that ρ_X is a metric on Θ_X. (This metric was first introduced by F. Hausdorff himself, and is thus know as the Hausdorff metric.*)*

b) Show that if $X = \mathbb{R}$ and d is its standard metric given by the absolute value, the restriction of the corresponding ρ_X to $\mathcal{J}$ given in (10) is precisely the metric given in (11).

2.2 The topology of uniform convergence

3 Continuous causal curves on a Lorentz manifold

Throughout these notes, we fix an n-dimensional (C^∞) spacetime (M,g) ($n \geq 2$). We also fix an auxiliary complete Riemannian metric h on M. We denote by I any arbitrary (i.e., closed, semi-closed or open) non-empty interval in $\mathbb{R}$, unless otherwise stated.

[4]It is worth pointing out that in defining the space of unparametrized curves we could just as well have regarded as equivalent curves which differ by composition with *any* homeomorphism, either increasing or decreasing. For technical reasons we choose not to do that here.

Recall that for any $p, q \in M$, we write $p \ll q$ [resp. $p < q$] to mean that there exists a *piecewise smooth* future-directed timelike [resp. causal] curve segment $\alpha : [a, b] \to M$ with $\alpha(a) = p$ and $\alpha(b) = q$. If $\alpha[a, b] \subset \mathcal{U}$ for some subset $\mathcal{U} \subset M$ and we wish to emphasize this, then we write $p \ll_{\mathcal{U}} q$ [resp. $p <_{\mathcal{U}} q$]. Piecewise smoothness, however, turns out to be too restrictive when considering spaces of curves, as we shall be interest in doing here. Our first goal, therefore, will be to extend the notion of causality to *continuous* curves. This is accomplished by any of the equivalent statements in the following proposition.

We deal only with *future-directed* causal curves here; the analogous past-directed versions are understood to obtain via time-duality.

Proposition 3.1 *For a continuous curve $\alpha : I \subset \mathbb{R} \to M$, the following statements are equivalent.*

i) For any $t_0 \in I$, there exist a convex normal neighborhood $\mathcal{U}(t_0) \subset M$ of $\alpha(t_0)$ and a number $\varepsilon_0 > 0$ such that $\alpha(I \cap (t_0 - \varepsilon_0, t_0 + \varepsilon_0)) \subset \mathcal{U}(t_0)$ and for any $s, t \in I \cap (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$,

$$s < t \Rightarrow \alpha(s) <_{\mathcal{U}(t_0)} \alpha(t).$$

ii) For any $t_0 \in I$, and for any convex normal neighborhood $\mathcal{U} \subset M$ of $\alpha(t_0)$, there exists a number $\varepsilon_0 > 0$ such that $\alpha(I \cap (t_0 - \varepsilon_0, t_0 + \varepsilon_0)) \subset \mathcal{U}$ and for any $s, t \in I \cap (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$,

$$s < t \Rightarrow \alpha(s) <_{\mathcal{U}} \alpha(t).$$

iii) For any convex normal neighborhood $\mathcal{U} \subset M$ and for any $s, t \in I$ such that $s < t$ and $\alpha[s, t] \subset \mathcal{U}$, we have $\alpha(s) <_{\mathcal{U}} \alpha(t)$.

Proof. $(i) \Rightarrow (ii)$
Fix $t_0 \in I$, a convex normal neighborhood $\mathcal{U}(t_0) \subset M$ of $\alpha(t_0)$ and a number $\varepsilon_0 > 0$ as in (i). Given any convex normal neighborhood $\mathcal{U} \subset M$ of $\alpha(t_0)$, pick a convex normal neighborhood $\mathcal{V} \subset \mathcal{U} \cap \mathcal{U}(t_0)$. By continuity there exists a number $0 < \varepsilon \leq \varepsilon_0$ such that $\alpha(I \cap (t_0 - \varepsilon, t_0 + \varepsilon)) \subset \mathcal{V}$. For any $s, t \in I \cap (t_0 - \varepsilon, t_0 + \varepsilon)$ with $s < t$, the radial geodesic in $\mathcal{V}$ from $\alpha(s)$ to $\alpha(t)$ coincides with the corresponding radial geodesic in $\mathcal{U}(t_0)$, and we have $\alpha(s) <_{\mathcal{U}(t_0)} \alpha(t)$ by assumption, which means that this geodesic is causal; since it also coincides with that in $\mathcal{U}$, we conclude that $\alpha(s) <_{\mathcal{U}} \alpha(t)$.
$(ii) \Rightarrow (iii)$
Fix any convex normal neighborhood $\mathcal{U} \subset M$ and any $s, t \in I$ such that $s < t$ and $\alpha[s, t] \subset \mathcal{U}$. Using (ii), for each $\lambda \in [s, t]$, pick a number $\varepsilon_\lambda > 0$ such that $\alpha(I \cap (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)) \subset \mathcal{U}$ and for any $\lambda', \lambda'' \in I \cap (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$

$$\lambda' < \lambda'' \Rightarrow \alpha(\lambda') <_{\mathcal{U}} \alpha(\lambda'').$$

Let $\varepsilon > 0$ be a *Lebesgue number* for the open cover $\{(\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)\}_{\lambda \in [s,t]}$ of the compact set $[s, t]$: for any set $X \subset [s, t]$ with diameter less than ε, there exists

$\lambda \in [s,t]$ for which $X \subset (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$. Choose any partition $\lambda_0 = s < \cdots < \lambda_k = t$ of the interval $[s,t]$ with $\max\{|\lambda_i - \lambda_{i-1}| : i = 1, \ldots, k\} < \varepsilon$. Thus, by construction,

$$\alpha(\lambda_{i-1}) <_{\mathcal{U}} \alpha(\lambda_i), \quad i = 1, \ldots, k.$$

By choosing, for each $i = 1, \ldots, k$, a piecewise smooth future-directed causal curve segment β_i in $\mathcal{U}$ from $\alpha(\lambda_{i-1})$ to $\alpha(\lambda_i)$, and concatenating them, we end up with a piecewise smooth causal curve segment β connecting $\alpha(s)$ and $\alpha(t)$ in $\mathcal{U}$, as desired.
$(iii) \Rightarrow (i)$ is immediate.

□

Definition 3.1 (Continuous causal curves) *A continuous map $\alpha : I \subset \mathbb{R} \to M$ is a* future-directed C^0 causal curve *if any (and hence all) of the statements $(i) - (iii)$ in Prop. 3.1 holds.*

Remark 3.1 Two observations must be borne in mind about Definition 3.1.

a) In any convex open set $\mathcal{U} \subset M$, whenever $p <_{\mathcal{U}} q$ we in particular must have $p \neq q$. Thus, *no causal curve can self-intersect inside a convex normal neighborhood.* In particular, *a future-directed C^0 causal curve $\alpha : I \subset \mathbb{R} \to M$ can never become constant along a non-empty subinterval $J \subset I$.*

b) It follows quite easily from the definition that if $\alpha : I \subset \mathbb{R} \to M$ is a future-directed C^0 causal curve, and $f : J \subset \mathbb{R} \to I \subset \mathbb{R}$ is an increasing homeomorphism, then $\alpha \circ f$ is also a future-directed C^0 causal curve.

Causality can be described just as well via continuous curves, as we see next.

Proposition 3.2 *For any $p, q \in M$, $p < q$ if and only if there exists a future-directed C^0 causal curve $\alpha : [a,b] \to M$ with $\alpha(a) = p$ and $\alpha(b) = q$.*

Proof. The "only if" part is immediate. For the converse, fix a future-directed C^0 causal curve $\alpha : [a,b] \to M$ with $\alpha(a) = p$ and $\alpha(b) = q$, and a finite open cover $\mathcal{U}_1, \ldots, \mathcal{U}_k$ of the compact set $\alpha[a,b]$ by convex normal neighborhoods. Renaming these sets if needed, we can choose a partition $a = t_0 < \cdots < t_k = b$ of the interval $[a,b]$ such that $\alpha[t_{i-1}, t_i] \subset \mathcal{U}_i$, and hence $\alpha(t_{i-1}) <_{\mathcal{U}_i} \alpha(t_i)$ for each $i \in \{1, \ldots, k\}$ by item (iii) in Prop. 3.1. By choosing, for each $i = 1, \ldots, k$, a piecewise smooth future-directed causal curve segment β_i in $\mathcal{U}_i$ from $\alpha(t_{i-1})$ to $\alpha(t_i)$, and concatenating them in succesion, we end up with a piecewise causal curve segment connecting p and q, as desired.

□

Corollary 3.1 *Suppose a function $f \in C^\infty(M)$ has past-directed timelike gradient ∇f everywhere. Then, for any future-directed C^0 causal curve $\alpha : I \subset \mathbb{R} \to M$, the real-valued continuous function $f \circ \alpha : I \to \mathbb{R}$ is strictly increasing.*

Proof. Let $t, s \in I$ with $s < t$. Then by Prop. 3.2 there exists a piecewise smooth, future-directed causal curve $\gamma : [a, b] \to M$ with $\gamma(a) = \alpha(s)$ and $\gamma(b) = \alpha(t)$. Since ∇f is past directed timelike and γ is future-drected we have $g((\nabla f) \circ \gamma, \gamma') > 0$. Hence,

$$\begin{aligned} f(\alpha(t)) - f(\alpha(s)) &= (f \circ \gamma)(b) - (f \circ \gamma)(a) \\ &= \int_a^b (f \circ \gamma)'(\lambda) d\lambda \\ &= \int_a^b g((\nabla f) \circ \gamma(\lambda), \gamma'(\lambda)) d\lambda > 0 \\ &\Rightarrow f(\alpha(t)) > f(\alpha(s)). \end{aligned}$$

□

In defining a topology on the space of causal curves, as well as in the proof of the Limit Curve Lemma below, the following technical result will be of crucial importance

Lemma 3.1 *At each $p \in M$, there exists a coordinate system $(U, \phi = (x^1, \ldots, x^n))$ of M and a constant $C > 0$ with the following properties.*

1) ∇x^1 is past-directed timelike on U.

2) Any future-directed C^0 causal curve $\alpha : [a, b] \to M$ whose image is contained in U is h-rectifiable, and the h-length $L_h(\alpha)$ of α satisfies

$$L_h(\alpha) \leq C|x^1 \circ \alpha(b) - x^1 \circ \alpha(a)|. \tag{12}$$

Proof. Fix $p \in M$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM. Let $(V, \phi = (x^1, \ldots, x^n))$ be a normal coordinate system at p, such that

$$\frac{\partial}{\partial x^i}(p) \equiv e_i, \quad i \in \{1, \ldots, n\}.$$

In particular, $g_{ij}(p) = \eta_{ij}$ with respect to this system, so $\nabla x^1(p) = -\partial/\partial x^1(p) = e_1$ is timelike, and by continuity, we can find an open set $U \ni p$ such that $\overline{U} \subset V$ is compact, $\phi(U) \subset \mathbb{R}^n$ is an Euclidean open ball with center $\phi(p)$, and $\nabla x^1|_U$ is timelike everywhere.

We now claim that we can further shrink U so that the flat Lorentz metric g_0 on U given by the line element

$$ds^2 = -2(dx^1)^2 + \sum_{i=2}^{n} (dx^i)^2 \tag{13}$$

satisfies

$$v \in TU \setminus \{0\} \text{ and } g(v, v) \leq 0 \Longrightarrow g_0(v, v) < 0.$$

Indeed, suppose this claim is false. Then we can find sequences $(q_k) \subset M$ and $v_k \in T_{q_k}M \setminus \{0\}$ with

$$q_k \to p, g(v_k, v_k) \leq 0 \text{ and } g_0(v_k, v_k) \geq 0.$$

We can assume without loss of generality that $(q_k) \subset V$ and $v_k = v_k^i \cdot \frac{\partial}{\partial x^i}\big|_{q_k}$ satisfies $\|(v_k^1, \dots, v_k^n)\| = 1$ for every $k \in \mathbb{N}$, where $\|\,.\,\|$ denotes the Euclidean norm. By the compactness of the Euclidean unit sphere, we can assume, up to passing to a subsequence, that $v_k^i \to v_0^i$, for each $i \in \{1, \dots, n\}$, where

$$\|(v_0^1, \dots, v_0^n)\| = 1. \tag{14}$$

If we put $v_0 := v_0^i \cdot \frac{\partial}{\partial x^i}\big|_p$ we conclude that $v_k \to v_0$ on TM. Therefore, $g(v_0, v_0) \leq 0$ and $g_0(v_0, v_0) \geq$ by continuity. Explicitly,

$$g(v_0, v_0) \leq 0 \Rightarrow \sum_{i=2}^{n} (v_0^i)^2 \leq (v_0^1)^2 \text{ and } g_0(v_0, v_0) \geq 0 \Rightarrow \sum_{i=2}^{n} (v_0^i)^2 \geq 2(v_0^1)^2,$$

which combined imply that $v_0 \equiv 0$, contradicting (14). This contradiction establishes the claim, and for the remainder of the proof we assume that U has been shrunk accordingly.

We next claim that exists a number $c > 0$ for which

$$\sqrt{h(v,v)} \leq c \cdot \|(v^1, \dots, v^n)\|, \quad \forall v = v^i \cdot \frac{\partial}{\partial x^i} \in TU. \tag{15}$$

For suppose not. Then, for every $k \in \mathbb{N}$ we could find a $q_k \in U$ and $v_k = v_k^i \cdot \frac{\partial}{\partial x^i}\big|_{q_k}$ such that

$$h(v_k, v_k) = h_{ij}(q_k) v_k^i v_k^j > k \cdot \|(v_k^1, \dots, v_k^n)\|^2. \tag{16}$$

In particular, the strict inequality in (16) implies that each v_k is nonzero, so the vectors

$$\hat{v}_k := \frac{(v_k^1, \dots, v_k^n)}{\|(v_k^1, \dots, v_k^n)\|}$$

again live on the Euclidean unit sphere. By the compactness of the latter and of $\overline{U}$, we can again assume up to passing to a subsequence that $q_k \to q \in \overline{U} \subset V$ and $\hat{v}_k \to \hat{v} = (\hat{v}^1, \dots, \hat{v}^n) \in \mathbb{S}^{n-1}$. Thus, on the one hand,

$$h_{ij}(q_k) \hat{v}_k^i \hat{v}_k^j \to h_{ij}(q) \hat{v}^i \hat{v}^j,$$

on the other hand (16) implies that $h_{ij}(q_k) \hat{v}_k^i \hat{v}_k^j \to +\infty$, a contradiction. Thus the claim is established.

Now, fix $c > 0$ such that (15) holds, and define $C = c\sqrt{3}$. To finalize the proof, we must show that the coordinate system $(U, \phi|_U)$ is the desired one. Let therefore $\alpha : [a, b] \to M$ be a future-directed C^0 causal curve such that $\alpha[a, b] \subset U$. Let

$$P = \{t_0 = a < \cdots t_m = b\}$$

be any partition of the interval $[a,b]$. Fix any $\ell \in \{1,\dots,m\}$. Firstly, note that by Prop. 3.2 applied to $(U, g|_U)$ viewed as a spacetime in irts own right we have $\alpha(t_{\ell-1}) <_U \alpha(t_\ell)$, and therefore, by the chosen properties of the metric (13) we have $\alpha(t_{\ell-1}) \ll_{g_0} \alpha(t_\ell)$ in U. Secondly, Corollary 3.1 also applied to $(U, g|_U)$ means that the function

$$f : s \in [t_{\ell-1}, t_\ell] \mapsto x^1 \circ \alpha(s) \in \mathbb{R}$$

is strictly increasing, and thus defines a homeomorphism onto its image $Imf =: [s_{\ell-1}, s_\ell]$. Write $z^j(s) := x^j((\alpha \circ f^{-1})(s))$ for each $s \in [s_{\ell-1}, s_\ell]$, $j \in \{1,\dots,n\}$, and note that $z^1(s) \equiv s$. Put also $z(s) = (z^1(s),\dots,z^n(s))$. Consider now the straight line segment in $\mathbb{R}^n$ given by

$$Z(s) = z(s_{\ell-1}) + \frac{s - s_{\ell-1}}{s_\ell - s_{\ell-1}} \cdot [z(s_\ell) - z(s_{\ell-1})] \quad s \in [s_{\ell-1}, s_\ell].$$

Now, it is easily verified that $Z(s_{\ell-1}) = \phi(\alpha(t_{\ell-1}))$ and $Z(s_\ell) = \phi(\alpha(t_\ell))$. Since $\phi(U)$ is an Euclidean ball, it is in particular convex, so that $Im(\phi^{-1} \circ Z) \subset \phi(U)$, and $\phi^{-1} \circ Z(s_{\ell-1}) \ll_{g_0} \phi^{-1} \circ Z(s_\ell)$. We now conclude that

$$\sum_{j=2}^{n} \left[\frac{z^j(s_\ell) - z^j(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right]^2 < 2 \Rightarrow \|Z(s_\ell) - Z(s_{\ell-1})\| < \sqrt{3}(s_\ell - s_{\ell-1}), \qquad (17)$$

where we have added $1 = \left[\frac{z^1(s_\ell) - z^1(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right]^2$ on both sides of the inquality on the left to get the implication.

Finally,

$$\begin{aligned}
d_h(\alpha(t_{\ell-1}), \alpha(t_\ell)) &\leq \int_{s_{\ell-1}}^{s_\ell} \sqrt{h((\phi^{-1} \circ Z)'(s), (\phi^{-1} \circ Z)'(s))}ds \\
&= \int_{s_{\ell-1}}^{s_\ell} \sqrt{h_{ij}((\phi^{-1} \circ Z)(s)) \left(\frac{z^i(s_\ell) - z^i(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right)\left(\frac{z^j(s_\ell) - z^j(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right)}ds \\
&\overset{(15)}{\leq} c\int_{s_{\ell-1}}^{s_\ell} \frac{\|Z(s_\ell) - Z(s_{\ell-1})\|}{s_\ell - s_{\ell-1}} \\
&\overset{(17)}{\leq} C(s_\ell - s_{\ell-1}) \equiv C(f(t_\ell) - f(t_{\ell-1})). \qquad (18)
\end{aligned}$$

Summing inequality (18) over ℓ, we have

$$\sum_{\ell=1}^{m} d_h(\alpha(t_{\ell-1}), \alpha(t_\ell)) \leq C \cdot \sum_{\ell=1}^{m} (x^1 \circ \alpha(t_\ell) - x^1 \circ \alpha(t_{\ell-1})) = C(x^1 \circ \alpha(b) - x^1 \circ \alpha(a)).$$

The supremum on the left-hand side of the previous inequality over all the partitions P of $[a,b]$ now yields

$$L_h(\alpha) \leq C(x^1 \circ \alpha(b) - x^1 \circ \alpha(a)).$$

In particular, α is h-rectifiable as claimed.

□

Lemma 3.1 has an important immediate consequence: the h-rectifiability of C^0 causal curves, and the existence of a well-defined h-arc length function for them.

Proposition 3.3 (Riemannian arc length) *Let $\alpha : I \to M$ be future-directed C^0 causal curve. For any $s, t \in I$ with $s \leq t$, $\alpha|_{[s,t]}$ is an h-rectifiable curve. In addition, for any fixed $t_0 \in I$, the h-arc length function $S_\alpha^{t_0} : I \to \mathbb{R}$ given by*

$$S_\alpha^{t_0}(t) := \begin{cases} L_h(\alpha|_{[t_0,t]}) & \text{if } t \geq t_0, \\ -L_h(\alpha|_{[t,t_0]}) & \text{if } t < t_0 \end{cases}$$

is a continuous strictly increasing function. Furthermore, if $I = (a, b)$, with $-\infty \leq a < b \leq +\infty$ and $\alpha|_{[t_0,b)}$ [resp. $\alpha|_{(a,t_0]}$] is right-inextendible [resp. left-inextendible], then $S_\alpha^{t_0}[t_0, b) = [0, +\infty)$ [resp. $S_\alpha^{t_0}(a, t_0] = (-\infty, 0]$].

Proof. Let any $t_1, t_2 \in I$ with $t_1 < t_2$ be given. We can cover the compact set $\alpha[t_1, t_2] \subset M$ with fiinitely many coordinate neighborhoods

$$(U_1, (x_1^1, \dots, x_1^n)), \dots, (U_k, (x_k^1, \dots, x_k^n))$$

as in Lemma 3.1, and pick a partition $s_0 = t_1 < \cdots < s_k = t_2$ of the interval $[t_1, t_2]$ for which $\alpha[s_{\ell-1}, s_\ell] \subset U_\ell$ for each $\ell \in \{1, \dots, k\}$. Thus, each $\alpha|_{[s_{\ell-1},s_\ell]}$ is h-rectifiable, whence we conclude that $\alpha|_{[t_1,t_2]}$ is.

Moreover, we can find numbers $C_1, \dots, C_k > 0$ for which

$$S_\alpha^{t_0}(s_\ell) - S_\alpha^{t_0}(s_{\ell-1}) \leq C_\ell |x_\ell^1(\alpha(s_\ell) - x_{\ell-1}^1(\alpha(s_{\ell-1}))|, \quad \forall \ell \in \{1, \dots, k\}.$$

(Cf. Eq.(12).) This shows that $S_\alpha^{t_0}$ is continuous, since t_1, t_2 and the partition were chosen entirely arbitrarily. Now,

$$S_\alpha^{t_0}(t_2) = S_\alpha^{t_0}(t_1) + L_h(\alpha|_{[t_1,t_2]}). \tag{19}$$

We can pick a convex normal neighborhood $\mathcal{U}$ of $\alpha(t_1)$ (say) and for some $\varepsilon > 0$ we have $t_1 + \varepsilon < t_2$ and $\alpha[t_1, t_1 + \varepsilon] \subset \mathcal{U}$, whence $\alpha(t_1) <_{\mathcal{U}} \alpha(t_1 + \varepsilon)$, and in particular we must have $\alpha(t_1) \neq \alpha(t_1 + \varepsilon)$, so

$$L_h(\alpha|_{[t_1,t_2]}) \geq d_h(\alpha(t_1), \alpha(t_1 + \varepsilon)) > 0,$$

and then (19) yields $S_\alpha^{t_0}(t_2) > S_\alpha^{t_0}(t_1)$. This shows that $S_\alpha^{t_0}$ is strictly increasing.

Finally, we assume now that $\alpha|_{[t_0,b)}$ is right-inextendible, since the left-inextendible case is analogous. We of course have $S_\alpha^{t_0}[t_0, b) \subset [0, +\infty)$, so we need only to show the opposite inclusion. Let $A > 0$. Right-inextendibility means that for some sequence $(t_k) \subset [t_0, b)$ with $t_k \to b$, the sequence $(\alpha(t_k)) \subset M$ does not converge. Now, due to the Hopf-Rinow theorem, (M, d_h) is a complete metric space satisfying the Heine-Borel property; hence the latter sequence is not d_h-bounded. But then, for large enough k we have $t_k > t_0$ and

$$S_\alpha^{t_0}(t_k) = L_h(\alpha|_{[t_0,t_k]}) \geq d_h(\alpha(t_0), \alpha(t_k)) > A.$$

Since $S_\alpha^{t_0}$ is continuous we must have $S_\alpha^{t_0}(t_A) = A$ for some (unique) $t_A \in [t_0, b)$ by the intermediate value property. This concludes the proof.

□

Proposition 3.3, in view of Remark 3.1 above, allows one to choose special parametrizations for future-directed C^0 causal curves, the *h-arc length (re)parametrization*, as follows. Given a future-directed C^0 causal curve $\alpha : I \to M$, pick a number $t_0 \in I$. The function $S_\alpha^{t_0}$ defined in Proposition 3.3 is a homeomorphism onto its image $J_\alpha^{t_0} := S_\alpha^{t_0}(I)$, thus

$$\alpha \circ (S_\alpha^{t_0})^{-1} : J_\alpha^{t_0} \to M$$

is also a future-directed C^0 causal curve, an h-arc length reparametrization of α. If α is equal to one of its h-length reparametrizations, then it is simply said to be *h-arc length parametrized.* It is easy to see that this happens if and only if

$$L_h(\alpha|_{[t,t']}) = t' - t, \quad \forall t, t' \in I, t < t'. \tag{20}$$

4 The Limit Curve Lemma

In this section we prove one of the most important technical results in Lorentzian geometry. The following fact is needed in its proof but is also of independent interest. This can be loosely stated as: "the uniform limit of a sequence of causal curves is causal". However, this holds with a caveat, namely that the curves in the sequence be parametrized with respect to h-arc length.

Proposition 4.1 *Let $(\alpha_k : [a,b] \to M)_{k\in\mathbb{N}}$ be a sequence of future-directed C^0 causal curves, all defined on the same compact interval $[a,b]$ and h-arc length parametrized. If this sequence converges d_h-uniformly to a map $\alpha : [a,b] \to M$, then α is also a future-directed C^0 causal curve (not necessarily h-arc length parametrized).*

Proof. It is a standard fact that α is a continuous map. Fix now $t_0 \in [a,b]$ and let $(U, (x^1, \dots, x^n))$ be a coordinate system at $\alpha(t_0)$ as in Lemma 3.1. Choose also a convex normal neighborhood $\mathcal{U} \ni \alpha(t_0)$ with $\mathcal{U} \subset U$. Let $t_1 < t_2$ in $[a,b]$ such that $\alpha[t_1,t_2] \subset \mathcal{U}$. Uniform convergence implies that there exists $k_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ with $k \geq k_0$ we have $\alpha_k[t_1,t_2] \subset \mathcal{U}$. Then, for any such k, $\alpha_k(t_1) <_{\mathcal{U}} \alpha_k(t_2)$, so the initial velocities of the radial geodesics $v_k := \overrightarrow{\alpha_k(t_1)\alpha_k(t_2)}$ from $\alpha_k(t_1)$ to $\alpha_k(t_2)$ are causal vectors. Since $\alpha_k(t_i) \to \alpha(t_i)$ for $i = 1,2$ we have that $v = \overrightarrow{\alpha(t_1)\alpha(t_2)}$ is either zero or causal. Let $C > 0$ such that Eq.(12) holds for each $k \geq k_0$; taking (20) into account, we then have

$$t_2 - t_1 = L_h(\alpha_k|_{[t_1,t_2]}) \leq C|x^1(\alpha_k(t_1)) - x^1(\alpha_k(t_2))|, \forall k \geq k_0.$$

Upon taking the limit $k \to +\infty$, we get

$$0 < t_2 - t_1 \leq C|x^1(\alpha(t_1)) - x^1(\alpha(t_2))| \Rightarrow \alpha(t_2) \neq \alpha(t_1),$$

and thus $v \neq 0$; that is, it is causal, whence $\alpha(t_1) <_{\mathcal{U}} \alpha(t_2)$. This concludes the proof.

□

Theorem 4.1 (Limit Curve lemma) *Let $I \subset \mathbb{R}$ be either $I = [0, +\infty)$ or $I = (-\infty, +\infty)$, and let $(\gamma_k : I \to M)_{k\in\mathbb{N}}$ be a sequence of future-directed C^0 causal curves parametrized by h-arc length, inextendible if $I = \mathbb{R}$, future-inextendible if $I = [0, +\infty)$. Suppose the sequence $(\gamma_k(0))_{k\in\mathbb{N}}$ has a limit point $p \in M$. Then, there exists a future-directed C^0 causal curve $\gamma : I \to M$ (not necessarily parametrized by h-arc length, but inextendible if $I = \mathbb{R}$, future-inextendible if $I = [0, +\infty)$) such that $\gamma(0) = p$, and for some subsequence $(\gamma_{k_i})_{i\in\mathbb{N}}$ we have $\gamma_{k_i}(0) \to p$ and*

$$\gamma_{k_i}|_C \to \gamma|_C \ d_h\text{-uniformly in each compact } C \subset I.$$

A analogous, time-dual result holds for a sequence of past-directed *C^0 causal curves.*

Proof. In order to fix ideas we take $I = \mathbb{R}$, since the case of the half-interval is entirely analogous. Start by picking any subsequence $(\gamma_{k_i})_{i\in\mathbb{N}}$ with $\gamma_{k_i}(0) \to p$. Since each curve γ_{k_i} is h-arc length parametrized, we have, for any $t < t' \in I$ (cf. Eq.(20)):

$$d_h(\gamma_{k_i}(t), \gamma_{k_i}(t')) \leq L_h(\gamma_{k_i}|_{[t,t']}) = |t - t'|, \quad \forall i \in \mathbb{N}. \tag{21}$$

Eq. (21) implies (i) that the collection $\{\gamma_{k_i}\}_{i\in\mathbb{N}}$ is equicontinuous and (ii) that the sequence $(\gamma_{k_i}(t))_{i\in\mathbb{N}}$ is bounded (and hence contained in a compact set by Hopf-Rinow) for each $t \in I$. (This is obtained by taking, say, $t' = 0$ in (21).) By the Arzelà-Ascoli theorem on $C_c(\mathbb{R}, M)$, passing to a further subsequence if necessary, we can conclude that there exists a continuous map $\gamma : \mathbb{R} \to M$ such that

$$\gamma_{k_i}|_C \to \gamma|_C \ d_h\text{-uniformly in each compact } C \subset I.$$

Since for any $t < t' \in I$, we have $\gamma_{k_i}|_[t, t'] \to \gamma|_[t, t']$ d_h-uniformly, Proposition 4.1 implies that $\gamma|_{[t,t']}$ is a future-directed C^0 causal curve. We conclude that γ itself is such.

Finally, we need to show that γ is inextendible. It is enough to show it is right-inextendible, since the other side is again entirely analogous. Assume, by way of contradiction, that there exists a right-endpoint $q \in M$. Let $(U, (x^1, \ldots, x^n))$ at q as given in Lemma 3.1. For some $t_0 \in \mathbb{R}$ we therefore have $\gamma[t_0, +\infty) \subset U$. Consider the open set

$$\mathcal{O} = \{r \in U \,:\, x^1(\gamma(t_0)) < x^1(r) < x^1(q)\}.$$

Pick any sequence $(t_m) \subset [t_0, +\infty)$ converging to $+\infty$. Then $\gamma(t_m) \to q$. Given any $t \in [t_0, +\infty)$, eventually $t_m > t$. But since x^1 strictly increases along γ, we then will have

$$x^1(\gamma(t)) < x^1(\gamma(t_m)) \to x^1(q) \Longrightarrow x^1(\gamma(t)) < x^1(q).$$

We conclude that for any $t > t_0$ we shall have $\gamma[t, +\infty) \subset \mathcal{O}$. Let

$$R := C|x^1(q) - x^1(\gamma(t_0))| > 0,$$

where $C > 0$ is such that Eq.(12) holds. Since $\gamma_{k_i}|_{[t_0+1,t_0+R+2]} \to \gamma|_{[t_0+1,t_0+R+2]}$ d_h-uniformly, for large enough i we have $\gamma_{k_i}([t_0+1, t_0+R+2]) \subset \mathcal{O}$. But then

$$\begin{aligned} R+1 &= L_h(\gamma_{k_i}|_{[t_0+1,t_0+R+2]}) \leq C|x^1(\gamma_{k_i}(t_0+R+2)) - x^1(\gamma_{k_i}(t_0+1))| \\ &\leq C|x^1(q) - x^1(\gamma(t_0)| \equiv R, \end{aligned}$$

an absurd.

□