Stone-Weierstrass Theorem

Theorem. Let X be a compact metric space and let $C^0(X, \mathbb{R})$ be the algebra of continuous real functions defined over X. Let $\mathcal{A}$ be a subalgebra of $C^0(X, \mathbb{R})$ for which the following conditions hold:

(1) $\forall x, y \in X, x \neq y, \exists f \in \mathcal{A} : f(x) \neq f(y)$

(2) $1 \in \mathcal{A}$

Then $\mathcal{A}$ is dense in $C^0(X, \mathbb{R})$.

This theorem is a generalization of the classical Weierstrass approximation theorem to general spaces.

Let $\bar{\mathcal{A}}$ denote the closure of $\mathcal{A}$ in $C^0(X, \mathbb{R})$ according to the uniform convergence topology. We want to show that, if conditions 1 and 2 are satisfied, then $\bar{\mathcal{A}} = C^0(X, \mathbb{R})$.

First, we shall show that, if $f \in \bar{\mathcal{A}}$, then $|f| \in \bar{\mathcal{A}}$. Since f is a continuous function on a compact space f must be bounded – there exists constants a and b such that $a \leq f \leq b$. By the Weierstrass approximation theorem, for every $\epsilon > 0$, there exists a polynomial such that $|P(x) - |x|| < \epsilon$ when $x \in [a, b]$. See the proof of the Weierstrass theorem for an elementary construction of P. Define $g : X \to \mathbb{R}$ by $g(x) = P(f(x))$. Since $\bar{\mathcal{A}}$ is an algebra, $g \in \bar{\mathcal{A}}$. For all $x \in X$, $|g(x) - |f(x)|| < \epsilon$. Since $\bar{\mathcal{A}}$ is closed under the uniform convergence topology, this implies that $|f| \in \bar{\mathcal{A}}$.

A corollary of the fact just proven is that if $f, g \in \bar{\mathcal{A}}$, then $\max(f, g) \in \bar{\mathcal{A}}$ and $\min(f, g) \in \bar{\mathcal{A}}$. The reason for this is that one can write

$$\max(a, b) = \frac{1}{2}\left(|a + b| + |a - b|\right)$$

$$\min(a, b) = \frac{1}{2}\left(|a + b| - |a - b|\right)$$

Second, we shall show that, for every $f \in C^0(X, \mathbb{R})$, every $x \in X$, and every $\epsilon > 0$, there exists $g_x \in \bar{\mathcal{A}}$ such that $g_x \leq f + \epsilon$ and $g(x) > f(x)$. By condition 1, if $y \neq x$, there exists a function $\tilde{h}_{xy} \in \mathcal{A}$ such that $\tilde{h}_{xy}(x) \neq \tilde{h}_{xy}(y)$. Define h_{xy} by $h_{xy}(z) = p\tilde{h}_{xy}(z) + q$, where the constants p and q have been chosen so that

$$h_{xy}(x) = f(x) + \epsilon/2$$

$$h_{xy}(y) = f(y) - \epsilon/2$$

By condition 2, $h_{xy} \in \mathcal{A}$. (Note: This is the only place in the proof where condition 2 is used, but it is crucial since, otherwise, it might not be possible to construct a function which takes arbitrary preassigned values at two distinct points of X. The necessity of condition 2 can be shown by a simple example: Suppose that $\mathcal{A}$ is the algebra of all continuous functions on f which vanish at a point $O \in X$. It is easy to see that this algebra satisfies all the hypotheses of the theorem except condition 2 and the conclusion of the theorem is false in this case.)

For every $y \neq x$, define the set U_{xy} as

$$U_{xy} = \{z \in X \mid h_{xy}(z) < f(z) + \epsilon\}$$

Since f and h_{xy} are continuous, U_{xy} is an open set. Because $x \in U_{xy}$ and $y \in U_{xy}$, $\{U_{xy} \mid y \in X \setminus \{x\}\}$ is an open cover of X. By the definition of a compact space, there must exist a finite subcover. In other words, there exists a finite subset $\{y_1, y_2, \dots y_n\} \subset X$ such that $X = \bigcup_{m=0}^{n} U_{xy_m}$. Define $g_x = \min(h_{xy_1}, h_{xy_2}, \dots h_{xy_n}$. By the corollary of the first part of the proof, $g_x \in \bar{\mathcal{A}}$. By construction, $g_x(x) = f(x) + \epsilon/2$ and $g < f + \epsilon$.

Third, we shall show that, for every $f \in C^0(X, \mathbb{R})$ and every $\epsilon > 0$, there exists a function $g \in \bar{\mathcal{A}}$ such that $f \leq g < f + \epsilon$. This will complete the proof becauase it implies that $\bar{\mathcal{A}} = C^0(X, \mathbb{R})$. For every $x \in X$, define the set V_x as

$$V_x = \{z \in X \mid g_x(z) > f(x)\}$$

where g_x is defined as before. Since f and g_x are continuous, V_x is an open set. Because $g_x(x) = f(x) + \epsilon/2 > f(x)$, $x \in V_x$. Hence $\{V_x \mid x\}$ is an open cover of X. By the definition of a compact space, there must exist a finite subcover. In other words, there exists a finite subset $\{x_1, x_2, \dots x_n\} \subset X$ such that $X = \bigcup_{m=0}^{n} V_{x_n}$. Define g as

$$g(z) = \max\{g_{x_1}(z), g_{x_2}(z), \dots g_{x_n}(z)\}$$

By the corollary of the first part of the proof, $g \in \bar{\mathcal{A}}$. By construction, $g > f$. Since $g_x < f + \epsilon$ for every $x \in X$, $g < f + \epsilon$.